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ON THE LINES OF FORCE OF A MAGNETIC FIELD

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As is well known, the question of plasma flow in a given magnetic field may be investigated to a certain approximation with the help of finding its lines of force. This explains the interest currently aroused by the problem of finding the lines of force of specific type-magnetic fields.

From the mathematical viewpoint we are confronted here with the necessity of investigating a system of three standard differential equations possessing an invariant measure. In spite of efforts on a number of mathematicians, the investigation of such types of systems did not lead so far to a somewhat satisfactory answer in the general case to the question of trajectory behavior for systems of that class. Only one case exists when a satisfactory answer can be provided, and that if the case of fields possessing a certain symmetry (in the sense of existence of a uni-parametric group of invariant transformations).

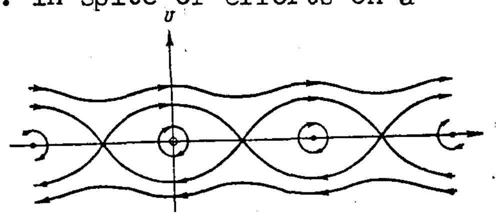

Fig. 1

In that case, the examined system of differential equations has a first integral, thanks to which the initial problem is reduced to the simpler problem of disposition of trajectories in a two-dimensional surface. The continous attempts by D. D. Birkhof to obtain an analogus result in the general case were unsuccessful, and as was shown by Siegel in 1954 [1], they could not have led to any other result.

The dispositon of the lines of force of near-symmetric magnetic fields have lately been investigated in a series of papers by A.I. Morozov, L. S. Solov'yev and others [2—6]. Making use of the averaging method, the above-named authors substituted the exact equations of lines of force by simpler, approximate equations which they investigated. As a result, they found that there are in the approximate equations regions with different line of force behavior, and they found the boundaries dividing these regions.

However, as I recently have shown [7], the application of the averaging method to finding the boundary separating regions with different line of force behavior, leads generally speaking to a qualitatively incorrect result. In reality, the boundary separating regions with different line of force behavior, has a much more complex form than it would appear from averaged equations for which such boundary disposition is generally impossible. Forestalling, I shall note that such an unusual boundary disposition, separating regions with different line of force behavior is one of the causes of appearnnce in the plasma of protuberances, leading to plasma instability, as is well known.

To illustrate the above-said, let us consider an irrotational magnetic field designated by the scalar potential

$$H_0 z + \psi(x, y, z),$$

where H_0 is a constant, and function $\psi(x, y, z)$ satisfies the Laplace equation, periodic along z with a period 2π and $\int_0^{2\pi} \psi(x, y, z)dz \equiv 0.$

The line of force equations for that field are

$$\frac{dx}{dz} = \frac{\psi'_x(x, y, z)}{H_0 + \psi'_z(x, y, z)}, \quad \frac{dy}{dz} = \frac{\psi'_y(x, y, z)}{H_0 + \psi'_z(x, y, z)}$$

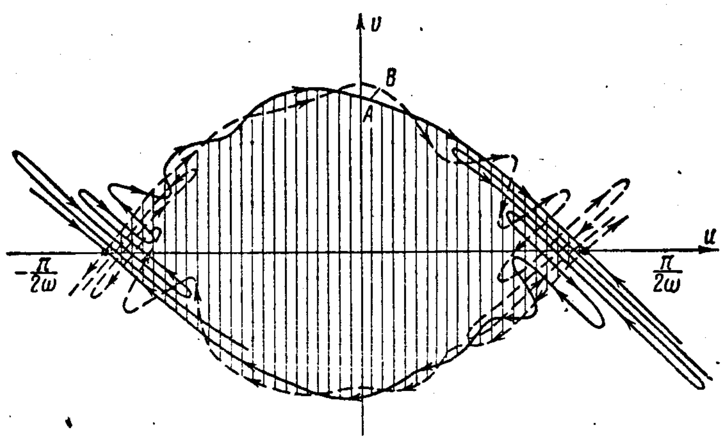

Fig. 2

which with the help of a conveniently selected substitution may me transformed into the form

$$\begin{aligned} \frac{du}{dz} &= \frac{1}{H_0^2} F'_v(u, v) + \frac{1}{H_0^3} f\left(u, v, z, \frac{1}{H_0}\right), \\ \frac{dv}{dz} &= -\frac{1}{H_0^2} F'_u(u, v) + \frac{1}{H_0^3} g\left(u, v, z\ \frac{1}{H_0}\right), \end{aligned} \tag{1}$$

where $$F(x, y) = \int_0^{2\pi} \varphi''_{xz}(x, y, z)\,\varphi'_y(x, y, z)\,dz = -\int_0^{2\pi} \varphi''_{yz}(x, y, z)\,\varphi'_x(x, y, z)\,dz,$$

while functions $f\left(u, v, z, \frac{1}{H_0}\right)$ and $g\left(u, v, z, \frac{1}{H_0}\right)$ are periodical along z with a period 2π.

The function (x, y, z), encoutered earlier, is obtained from (x, y, z) in the following manner: Let us assume

$$\psi(x, y, z) = \sum_{n \neq 0} \psi_n(x, y)\, e^{inz};$$

then

$$\varphi(x, y, z) = -i \sum_{n \neq 0} \frac{1}{n} \psi_n(x, y)\, e^{inz}.$$

Let us now assume

$$\psi(x, y, z) = \sin \omega x \,\mathrm{sh}\, \lambda y \sin z + \cos \omega x \,\mathrm{ch}\, \lambda y \cos z,$$

where $\lambda^2 = \omega^2 + 1;$ then $F(u, v) = \omega\lambda\pi(\mathrm{sh}^2 \lambda v + \sin^2 \omega u)$ and consequently the disposition of the system's trajectories

$$\frac{du}{dz} = \frac{1}{H_0^2} F'_v(u, v), \quad \frac{dv}{dz} = -\frac{1}{H_0^2} F'_u(u, v) \qquad (2)$$

will have the form represented in Fig. 1*.

On the other hand, following the ideas expounded in reference [8] one may determine for the system (1) a certain analogue of the separatrix playing for the type (1) systems the same role as the usual separatrix for a system of autonomous differential equations.

Taking advantage of the remark in [7], one may show that in order that the disposition of cross sections of that separatrix' analogue by the plane $z = z_0$ qualitatively coincide with the representation in Fig. 1 it is necessary that the functions $f\left(u, v, z, \frac{1}{H_0}\right)$ and $g\left(u, v, z, \frac{1}{H_0}\right)$ satisfy a certain infinite number of functional conditions.

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*) Let us note, by the way, that contrary to authors' assertions, the trajectory disposition of the system (2) for the potential

$$\psi(x, y, z) = a_1 \,\mathrm{ch} q_1 y \sin(kz - p_1 x) + a_2 \,\mathrm{sh} q_2 y \sin(kz - p_2 x)$$

proposed in reference [5] will have a shape having little in common with that represented in Fig. 1.

A direct computation shows that in the considered example we cannot even satisfy the first of these conditions, generally speaking. The consequence is that the cross section of system's (1) separatrix by the plane $z = z_0$ will have the form indicated in Fig. 2. The solid line indicates the cross section of system's (1) separatrix, the dotted line — the cross section of the separatrix of the sytem obtained from (1) by substituting z by $-z$. The arrows indicate in what direction the cross section points are displaced at $\Delta z = 2\pi$ shift along z. Hence it follows, that part of the solutions that come out of the shaded region of Fig. 2 at $z = z_0$, will leave that region with the increase of z, while among solutions, leaving this region, there are some which will remain in it any length of time.

It is easy to see that the share of solutions leaving the shaded region of Fig. 2, will depend on the value of the cut AB. It may be shown that for $H_0 \to \infty$, that value will tend to zero as $e^{-\alpha H_0^2}$, where $\alpha > 0$, and consequently the value of the cut AB will be rather small at comparatively small values H_0. It is possible, by appropriately selecting $\psi(x, y, z)$ to obtain in principle that for $H_0 \to \infty$ the value of AB approach zero, for example, as $e^{-\gamma H_0^4}$, where $\gamma > 0$.

It must be noted that such optimistic situation as that considered in our example, will by far not always take place. Let us consider for example the disposition of the lines of force of a three-threaded helical field, perturbed by a corrugated field [6]. This field may be given by the following scalar potential:

$$\psi(r, \varphi, z) = H_0 z + \frac{h_3}{\alpha} I_3(3\alpha r) \sin 3(\varphi - \alpha z) + \frac{h_0}{\alpha} I_0(k\alpha r) \sin k\alpha z,$$

where $I_0(r)$ and $I_3(r)$ are Bessel functions of the zero and third order from an imaginary argument. The lines of force for that field may be written in the following form:

$$\frac{dr}{dz} = \frac{H_r}{H_z} = \frac{\psi_r'}{\psi_z'} = \frac{3h_3 I_3'(3\alpha r)\sin 3\theta + kh_0 I_0'(k\alpha r)\sin k\alpha z}{H_0 - 3h_3 I_3(3\alpha r)\cos 3\theta + kh_0 I_0(k\alpha r)\cos k\alpha z},$$

$$\frac{d\theta}{dz} = \frac{\frac{1}{r}H_\varphi - \alpha H_z}{H_z} = \frac{\frac{1}{r^2}\psi_\varphi' - \alpha\psi_z'}{\psi_z'} =$$

$$= \frac{-\alpha H_0 + \frac{h_3}{r}\frac{\partial}{\partial r}(rI_3'(3\alpha r))\cos 3\theta - \frac{h_0}{r}\frac{\partial}{\partial r}(rI_0'(k\alpha r))\cos k\alpha z}{H_0 - 3h_3 I_3(3\alpha r)\cos 3\theta + kh_0 I_0(k\alpha r)\cos k\alpha z}, \qquad (3)$$

where $\theta = \varphi - \alpha z$. At $h_0 = 0$, the system (3) has the first integral

$$\frac{H_0 \alpha r^2}{2} - rh_3 I_3'(3\alpha r)\cos 3\theta = \text{const},$$

according to which the disposition of the trajectories of the system (3) at $h_0 = 0$ has the form indicated in Fig. 3.

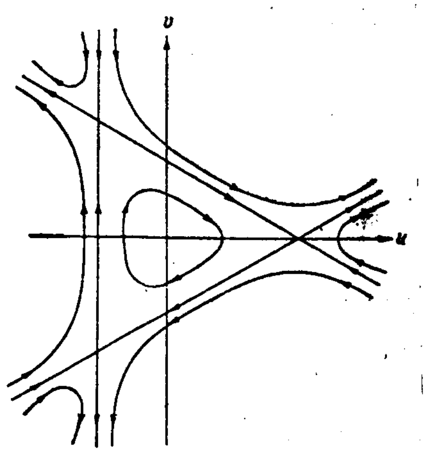

Fig. 3

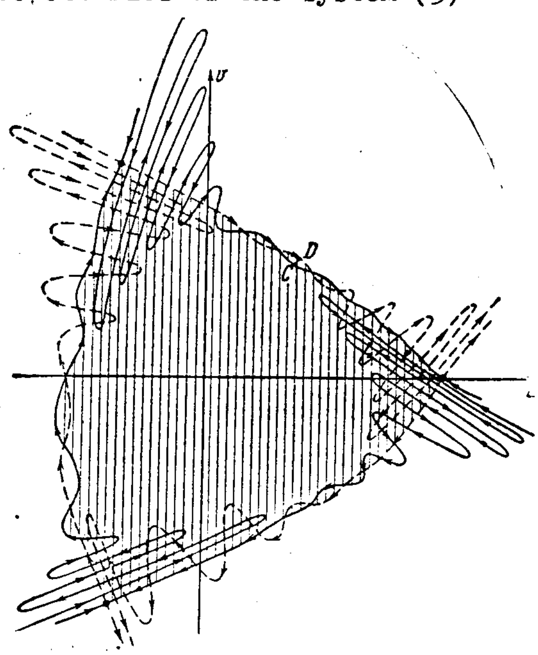

Fig. 4

On the other hand, using the results of the note [9], it is easy to establish that the cross section of system's (3) separatrix by the plane $z = z_0$ has the shape indicated in Fig.4, while for $h_0/H_0 \longrightarrow 0$, the value of the cut CD will approach zero as the first power of the qnatity h_0/H_0 (it is assumed here that the ratio h_3/H_0 is fixed).

As in the preceding example, there are among solutions coming out of the shaded region of Fig.4, some that will remain inside it as long as may be disired. That is why any attempt to find the boundary separating the solution of the system (3), remaining a long time in the shaded region of Fig.4, from the solutions coming out of it, with the help of numerical integration of the system (3) over a small interval of z variation, can hardly be expected to provide a satisfactory result, for it is far from clear whether or not if only all the solutions of system (3) coming out of the lobe regions will remain in the named region [5, 6].

It must be noted that basing ourselves on the results of the remark [9], we may construct examples of potentials, similar to the above-considered one, for which the value of the cut CD will have any assigned in advance order relative to the perturbation parameter. The selection of such fields for plasma retention may significantly increase the lifetime of plasma.

In conclusion I avail myself of this opportunity to express my gratitude to C. V. Fomin for his interest in the present work.

****** E N D ******

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